This is an AoPSWiki Word of the Week for June 6-12 Contents 1 Introduction & Formulas 2 Proof 3 Examples & Problems 4 See also The Angle bisector theorem states that given triangle and angle bisector AD, where D is on side BC, then . It follows that . Likewise, the converse of this theorem holds as well. Further by combining with Stewart's theorem it can be shown that Proof By the Law of Sines on and , First, because is an angle bisector, we know that and thus , so the denominators are equal. Second, we observe that and . Therefore, , so the numerators are equal. It then follows that Examples & Problems Let ABC be a triangle with angle bisector AD with D on line segment BC. If and , find AB and AC. Solution: By the angle bisector theorem, or . Plugging this into and solving for AC gives . We can plug this back in to find . In triangle ABC, let P be a point on BC and let . Find the value of . Solution: First, we notice that . Thus, AP is the angle bisector of angle A, making our answer 0. Part (b), 1959 IMO Problems/Problem 5 See also Angle bisector Geometry Stewart's theorem In \(\triangle{ABC}\), \(\lvert\overline{AB}\rvert=10, \lvert\overline{BC}\rvert=8, \lvert\overline{AC}\rvert=12\). Let \(D\) be a point on side \(\overline{AB}\) such that \(\overline{CD}\) bisects \(\angle C\). Then what is the length of \(\overline{AD}\)? Let \(\lvert\overline{AB}\rvert=c, \lvert\overline{BC}\rvert=a, \lvert\overline{AC}\rvert=b, \lvert\overline{AD}\rvert=y, \lvert\overline{BD}\rvert=x\), then we are now looking for \(y.\) Using the angle bisector theorem, \(\frac{y}{b}=\frac{x}{a}\implies \frac{y}{12}=\frac{x}{8}. \qquad (1) \) Since \(x+y=c\) or \(x=c-y,\) plugging this into \((1)\), we have \[\begin{align} \frac{y}{12}&=\frac{10-y}{8}\\\\ \Rightarrow y&=6. \ _\square \end{align}\] This next example is the same as the previous, but we are instead solving for the length \(e\) of the angle bisector \(\overline{CD}\). In \(\triangle{ABC}\), \(\lvert\overline{AB}\rvert=10, \lvert\overline{BC}\rvert=8, \lvert\overline{AC}\rvert=12\). Let \(D\) be a point on side \(\overline{AB}\) such that \(\overline{CD}\) bisects \(\angle C\), then what is the length of \(\overline{CD}\)? In our previous example, we already found \(\lvert\overline{AD}\rvert=6\) and \(\lvert\overline{BD}\rvert=4\). Thus, \[\begin{align} e &=\sqrt{ab-xy}\\ &= \sqrt{(8)(12)-(4)(6)}\\ &=6\sqrt{2}. \ _\square \end{align}\] We are given a triangle with the following property: one of its angles is quadrisected (divided into four equal angles) by the height, the angle bisector, and the median from that vertex. Find the measure of the quadrisected angle. The base is partitioned into four segments in the ratio \(x : x : y : 2x + y\). Suppose the length of the left-hand side of the triangle is \(1\). Then the length of the angle bisector is also \(1\). Applying the angle bisector theorem to the large triangle, we see that the length of the right-hand side is \[\dfrac{2x+2y}{2x}=1+\dfrac{y}{x}.\] But if we apply the angle bisector theorem to the left half of the triangle, we obtain \(\frac{2x+y}{y}= 1+\dfrac{2x}{y}\) for the same length. Therefore, \[\dfrac{y}{x}=\dfrac{2x}{y} \implies x:y=1:\sqrt2.\] Now apply the angle bisector theorem a third time to the right triangle formed by the altitude and the median. The segments in the base are in the ratio \(x:y=1:\sqrt2\), so the altitude and the median form the same ratio. As this is a right triangle, it must be a 45\(^\circ\)-45\(^\circ\)-90\(^\circ\) triangle. So the quadrisected angle is right. \(_\square\) Written By Gurudath Last Modified 25-01-2023 A ray that divides a given angle into two equal angles is known as an angle bisector. A bisector is a line dividing something into two equal parts. In geometry, a bisector is applied to the line segments and angles. A line that passes through the midpoint of a line segment is the bisector of the line segment. In this article, let us discuss the definition of an angle bisector, the construction of an angle bisector in a triangle, and solve some examples based on the angle bisector. An angle bisector is a ray that divides a given angle into two angles with equal measures. We usually divide an angle in a triangle by a line or ray, which is considered an angle bisector. Bisecting an angle means drawing a ray in the interior of the angle, with its initial point at the vertex of the angle such that it divides the angle into two equal parts. Example: Consider an angle \(\angle A B C=80^{\circ}\). An angle bisector divides it into two equal angles of \(40^{\circ}\). To draw a ray \(AX\) bisecting a given angle \(\angle BAC\), follow the below steps. With centre \(A\) and any convenient radius, draw an arc cutting \(AB\) and \(AC\) at \(P\) and \(Q\), respectively. With centre \(P\) and radius more than \(\frac{1}{2} P Q_{,}\) draw an arc. With centre \(Q\) and the same radius, as in the above step, draw another arc intersecting the arc in step \(2\) at \(R\). Join \(AR\) and produce it to any point \(X\). The ray \(AX\) is the required bisector of \(\angle B A C \). Verification: Measure \(\angle B A X\) and \(\angle C A X\). We will find that \(\angle B A X=\angle C A X\). A line segment that bisects one of the vertex angles of a triangle and ends up on the corresponding side of a triangle is known as the angle bisector of a triangle. There are three-angle bisectors in a triangle. The bisectors of a triangle meet in a single point called the incentre. This point is always inside the triangle. In the above triangle \(A B C, \alpha, \beta, \gamma\) are the angles of vertices \(A, B\) and \(C\) respectively. The angle bisector \(l_{a}\) divides angle \(\alpha\) into two equal parts \(\frac{\alpha}{2}\) and \(\frac{\alpha}{2}\), bisector \(l_{b}\) divides angle \(\beta\) into two equal parts \(\frac{\beta}{2}\) and \(\frac{\beta}{2}\) respectively, and bisector \(l_{c}\) divides the angle \(\gamma\) into two equal parts \(\frac{\gamma}{2}\) and \(\frac{\gamma}{2}\). Any point on the bisector of an angle is equidistant from the sides of the angle. In a triangle, the angle bisector divides the opposite side in the ratio of the adjacent sides that contain the angle. The bisector of a triangle that divides the opposite side internally in the ratio of corresponding sides containing angles is known as the internal bisector of an angle of a triangle. The bisector of a triangle that divides the opposite side externally in the ratio of corresponding sides containing angles is known as the external bisector of an angle of a triangle. In the below two theorems, we will learn that the internal bisector of an angle of a triangle divides the opposite side in the ratio of the sides containing the angle and vice-versa. Theorem 1: The internal angle bisector of a triangle divides the opposite side internally in the ratio of the sides containing the angle. Given: In \(\triangle A B C, A D\) is the internal bisector of \(\angle A\) and meets \(B C\) in \(D\). To prove: \(\frac{B D}{D C}=\frac{A B}{A C}\) Construction: Draw \(C E || D A\) to meet \(B A\) produced to \(E\). Proof: \(C E || D A\) and \(A C\) cuts them. Therefore, \(\angle 2=\angle 3….(i)\) (Alternate angles)and \(\angle 1=\angle 4….(ii)\) (Corresponding angles)But, \(\angle 1=\angle 2\) (\(A D\) is the bisector of \(\angle A\))From \((i)\) and \((ii)\), we get\(\angle 3=\angle 4\) Thus, in \(\Delta ACE,\) we have\(\angle 3 = \angle 4\)\(\Rightarrow AE = AC……\left({iii} \right)\) [Sides opposite to equal angles are equal]Now, in \(\Delta BCE,\) we have\(DA||CE\)\(\Rightarrow \frac{{BD}}{{DC}} = \frac{{BA}}{{AE}}\) [Basic proportionality theorem] \(\Rightarrow \frac{{BD}}{{DC}} = \frac{{AB}}{{AC}}\) [From \(\left({iii} \right)\)] Hence, proved. Theorem 2: In a triangle \(A B C\), if \(D\) is a point on \(B C\) such that \(\frac{B D}{D C}=\frac{A B}{A C}\), then \(A D\) is the bisector of \(\angle A\). Given: In \(\triangle A B C\), in which \(D\) is a point on \(BC\) such that, \(\frac{B D}{D C}=\frac{A B}{A C}\) Construction: Produce \(B A\) to \(E\) such that \(A E=A C\). Join \(E C\). Proof: In \(\triangle A C E\), we have\(\angle E=\angle C \) [By construction]\(\Rightarrow \angle 3=\angle 4….(1)\) Now, \(\frac{B D}{D C}=\frac{A B}{A C}\)\(\Rightarrow \frac{B D}{D C}=\frac{B A}{A E} \quad[A C=A E]\) Thus, in \(\triangle B C E\), we have \(\frac{B D}{D C}=\frac{A B}{A E}\) Therefore, by the converse of the Basic proportionality theorem, we have\(D A || C E\)\(\Rightarrow \angle 1=\angle 4 \ldots(ii)\) [Corresponding angles] and, \(\angle 2=\angle 3…(iii)\) [Alternate angles] But, \(\angle 3=\angle 4\) from \((i)\)\(\Rightarrow \angle 1=\angle 2\) [from \((ii)\) and \((iii)\)] Hence, \(A D\) is the bisector of \(\angle A\). In the following theorem, we shall prove that the bisector of the exterior of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle. Theorem 3: The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle. Given: In \(\triangle A B C, A D\) is the bisector of exterior angle \(\angle A\) and intersects \(B C\) produced in \(D\) To prove: \(\frac{B D}{C D}=\frac{A B}{A C}\) Construction: Draw \(CE || DA\) meeting \(AB\) in \(E\). Proof: Since \(CE || DA\) and \(AC\) intersects them. Therefore, \(\angle 1=\angle 3…(i)\)Also, \(CE || DA\) and \(BK\) intersects them. Now, in \(\triangle B A D\), we have\(E C || A D\)Therefore, \(\frac{B D}{C D}=\frac{B A}{E A}\) [Using corollary of Basic proportionality theorem]\(\Rightarrow \frac{B D}{C D}=\frac{A B}{A E}\)\(\Rightarrow \frac{B D}{C D}=\frac{A B}{A C}\) [\(A E=A C\) from \((iii)\)] Hence, proved. Let us understand the concept through some Angle Bisector of Triangle practices problem. Q.1. In the below figure, \(AD\) is the bisector of \(\angle A\). If \(B D=4 \mathrm{~cm}\) and \(D C=3 \mathrm{~cm}\) Find \(AC\). Ans: In \(\triangle A B C, A D\) is the bisector of \(\angle A\).Therefore, \(\frac{B D}{C D}=\frac{A B}{A C}\)\(\Rightarrow \frac{4}{3}=\frac{6}{A C}\)\(\Rightarrow 4 A C=18\)\(\Rightarrow A C=\frac{18}{4}=\frac{9}{2}=4.5 \mathrm{~cm}\) Q.2. In the below figure, \(AD\) is the bisector of \(\angle B A C\). If \(A B=10 \mathrm{~cm}, A C=14 \mathrm{~cm}\) and \(B C=6 \mathrm{~cm}\), find \(B D\) and \(D C\).Ans: Let \(B D=x \mathrm{~cm}\). Then, \(D C=(6-x) \mathrm{cm}\) Since \(AD\) is the bisector of \(\angle A\).\(\frac{A B}{A C}=\frac{B D}{D C}\)\(\Rightarrow \frac{10}{14}=\frac{B D}{D C}\)\(\Rightarrow \frac{10}{14}=\frac{x}{6-x}\)\(\Rightarrow 30-5 x=7 x\)\(\Rightarrow 12 x=30\)\(\Rightarrow x=\frac{5}{2}=2.5 \mathrm{~cm}\)\(\Rightarrow B D=2.5 \mathrm{~cm}\) and \(D C=6-x=6-2.5=3.5 \mathrm{~cm}\) Q.3. If the diagonal \(BD\) of a quadrilateral \(ABCD\) bisect both \(\angle B\) and \(\angle D\), show that \(\frac{A B}{A C}=\frac{A D}{C D}\).Ans: Given: A quadrilateral \(ABCD\) in which the diagonal \(BD\) bisects \(\angle B\) and \(\angle D\).To prove: \(\frac{A B}{A C}=\frac{A D}{C D}\)Construction: Join \(AC\) intersecting \(BD\) in \(O\). Proof: In \(\Delta A B C, B O\) is the bisector of \(\angle B\).Therefore, \(\frac{A O}{O C}=\frac{B A}{B C}\)\(\Rightarrow \frac{O A}{O C}=\frac{A B}{B C} \quad \ldots \ldots(\mathrm{i})\)In \(\Delta A D C, D O\) is the bisector of \(\angle D\).Therefore, \(\frac{A O}{O C}=\frac{A D}{C D}\)\(\Rightarrow \frac{O A}{O C}=\frac{A D}{C D} \quad \ldots \ldots(\mathrm{ii})\)From \((i)\) and \((ii)\), we get \(\frac{A B}{B C}=\frac{A D}{C D}\) Q.4. If the bisector of an angle of a triangle bisects the opposite side, prove that the triangle is isosceles.Ans: Given: In \(\Delta ABC,\) the bisector \(AD\) of \(\angle A\) bisects the side \(BC.\)To prove: \(AB = AC\)Proof: In \(\Delta ABC,\,AD\) is the bisector of \(\angle A.\)Therefore, \(\frac{{AB}}{{AC}} = \frac{{BD}}{{DC}}\)\(\Rightarrow \frac{{AB}}{{AC}} = 1\) (\(D\) is the midpoint of \(BC.\) So, \(BD = DC\))\(\Rightarrow AB = AC\)Hence, \(\Delta ABC\) is an isosceles triangle. Q.5. In \(\triangle A B C\), the bisector of \(\angle B\) meets \(AC\) at \(D\). A line \(PQ || AC\) meets \(AB, BC\) and \(B D\) at \(P, Q\) and \(R\) respectively. Show that(i) \(P R . B Q=Q R . B P\)(ii) \(A B \times C Q=B C \times A P\)Ans:Given: In \(\triangle A B C\), the bisector of \(\angle B\) and line \(PQ ||AC\) meets \(A, B C\) and \(B D\) at \(P, Q\) and \(R\) respectively. To prove:(i) \(P R . B Q=Q R . B P\)(ii) \(A B \times C Q=B C \times A P\) Proof:(i) In \(\triangle B Q P, B R\) is the bisector of \(\angle B\).Therefore, \(\frac{B Q}{B P}=\frac{Q R}{P R}\)\(\Rightarrow B Q \cdot P R=B P \cdot Q R\)\(\Rightarrow P R . B Q=Q R . B P\)(ii) In \(\triangle A B C\), we have \(PQ || AC\)\(\Rightarrow \frac{A B}{A P}=\frac{C B}{C Q}\) (Thales theorem)\(\Rightarrow A B \times C Q=B C \times A P\)The Angle Bisector of Triangle facts can be understood from the above-solved questions. We studied the definition of angle bisectors, construction of angle bisectors, the definition of angle bisector of a triangle, properties of angle bisector of a triangle and theorems on internal and external angle bisectors of a triangle. Also, we solved some example problems based on the angle bisector of a triangle to revise the learned concept. Q.1. Does angle bisector cut an angle in half?Ans: Yes, the angle bisector cut an angle in half.Example: Consider an angle \(\angle A B C=60^{\circ}\). An angle bisector divides it into two equal angles of \(30^{\circ}\). Q.2. Explain the angle bisector of a triangle with an example?Ans: A line segment that bisects one of the vertex angles of a triangle and ends up on the corresponding side of a triangle is known as the angle bisector of a triangle.There are three angle bisectors in a triangle. The three angle bisectors of a triangle meet in a single point called the incentre. This point is always inside the triangle.Example: Consider an angle \(\angle A B C=90^{\circ}\). An angle bisector divides it into two equal angles of \(45^{\circ}\). Q.3. What is an angle bisector, and what does it do?Ans: A ray that divides a given angle into two angles with equal measures is called an angle bisector. So, it divides an angle into equal halves.Example: Consider an angle \(\angle A B C=120^{\circ}\). An angle bisector divides it into two equal angles of \(60^{\circ}\). Q.4. What is the property of an angle bisector of a triangle?Ans: The properties of the angle bisector of a triangle are:1. Any point on the bisector of an angle is equidistant from the sides of the angle.2. In a triangle, the angle bisector divides the opposite side in the ratio of the adjacent sides. Q.5. Does the angle bisector go through the midpoint of the side opposite to it?Ans: A bisector of a line segment will pass through the midpoint of the line segment. To bisect an angle means to divide it into two equal parts. However, an angle bisector need not bisect the side opposite to it. So, we cannot say that the angle bisector will go through the midpoint of the side opposite to it. Q.6. Which is the best definition for angle bisector?Ans: The line that passes through the vertex of an angle dividing it into two equal parts is known as the angle bisector. We hope the article on Angle Bisector of Triangle will help solve the doubts. Readers can take Angle Bisector of Triangle notes from the article to revise the concept quickly. Geometrical theorem relating the lengths of two segments that divide a triangle The theorem states for any triangle ∠ DAB and ∠ DAC where AD is a bisector, then | B D | : | C D | = | A B | : | A C | . {\displaystyle |BD|:|CD|=|AB|:|AC|.} In geometry, the angle bisector theorem is concerned with the relative lengths of the two segments that a triangle's side is divided into by a line that bisects the opposite angle. It equates their relative lengths to the relative lengths of the other two sides of the triangle. Consider a triangle △ABC. Let the angle bisector of angle ∠ A intersect side BC at a point D between B and C. The angle bisector theorem states that the ratio of the length of the line segment BD to the length of segment CD is equal to the ratio of the length of side AB to the length of side AC: | B D | | C D | = | A B | | A C | , {\displaystyle {\frac {|BD|}{|CD|}}={\frac {|AB|}{|AC|}},} and conversely, if a point D on the side BC of △ABC divides BC in the same ratio as the sides AB and AC, then AD is the angle bisector of angle ∠ A. The generalized angle bisector theorem (which is not necessarily an angle bisector theorem, since the angle ∠ A is not necessarily bisected into equal parts) states that if D lies on the line BC, then | B D | | C D | = | A B | sin ∠ D A B | A C | sin ∠ D A C . {\displaystyle {\frac {|BD|}{|CD|}}={\frac {|AB|\sin \angle DAB}{|AC|\sin \angle DAC}}.} This reduces to the previous version if AD is the bisector of ∠ BAC. When D is external to the segment BC, directed line segments and directed angles must be used in the calculation. The angle bisector theorem is commonly used when the angle bisectors and side lengths are known. It can be used in a calculation or in a proof. An immediate consequence of the theorem is that the angle bisector of the vertex angle of an isosceles triangle will also bisect the opposite side. There exist many different ways of proving the angle bisector theorem. A few of them are shown below. Animated illustration of the angle bisector theorem. As shown in the accompanying animation, the theorem can be proved using similar triangles. In the version illustrated here, the triangle △ A B C {\displaystyle \triangle ABC} gets reflected across a line that is perpendicular to the angle bisector A D {\displaystyle AD} , resulting in the triangle △ A B 2 C 2 {\displaystyle \triangle AB_{2}C_{2}} with bisector A D 2 {\displaystyle AD_{2}} . The fact that the bisection-produced angles ∠ B A D {\displaystyle \angle BAD} and ∠ C A D {\displaystyle \angle CAD} are equal means that B A C 2 {\displaystyle BAC_{2}} and C A B 2 {\displaystyle CAB_{2}} are straight lines. This allows the construction of triangle △ C 2 B C {\displaystyle \triangle C_{2}BC} that is similar to △ A B D {\displaystyle \triangle ABD} . Because the ratios between corresponding sides of similar triangles are all equal, it follows that | A B | / | A C 2 | = | B D | / | C D | {\displaystyle |AB|/|AC_{2}|=|BD|/|CD|} . However, A C 2 {\displaystyle AC_{2}} was constructed as a reflection of the line A C {\displaystyle AC} , and so those two lines are of equal length. Therefore, | A B | / | A C | = | B D | / | C D | {\displaystyle |AB|/|AC|=|BD|/|CD|} , yielding the result stated by the theorem. In the above diagram, use the law of sines on triangles △ABD and △ACD: | A B | | B D | = sin ∠ A D B sin ∠ D A B {\displaystyle {\frac {|AB|}{|BD|}}={\frac {\sin \angle ADB}{\sin \angle DAB}}} 1 | A C | | C D | = sin ∠ A D C sin ∠ D A C {\displaystyle {\frac {|AC|}{|CD|}}={\frac {\sin \angle ADC}{\sin \angle DAC}}} 2 Angles ∠ ADB and ∠ ADC form a linear pair, that is, they are adjacent supplementary angles. Since supplementary angles have equal sines, sin ∠ A D B = sin ∠ A D C . {\displaystyle {\sin \angle ADB}={\sin \angle ADC}.} Angles ∠ DAB and ∠ DAC are equal. Therefore, the right hand sides of equations (1) and (2) are equal, so their left hand sides must also be equal. | B D | | C D | = | A B | | A C | , {\displaystyle {\frac {|BD|}{|CD|}}={\frac {|AB|}{|AC|}},} which is the angle bisector theorem. If angles ∠ DAB, ∠ DAC are unequal, equations (1) and (2) can be re-written as: | A B | | B D | sin ∠ D A B = sin ∠ A D B , {\displaystyle {{\frac {|AB|}{|BD|}}\sin \angle DAB=\sin \angle ADB},} | A C | | C D | sin ∠ D A C = sin ∠ A D C . {\displaystyle {{\frac {|AC|}{|CD|}}\sin \angle DAC=\sin \angle ADC}.} Angles ∠ ADB, ∠ ADC are still supplementary, so the right hand sides of these equations are still equal, so we obtain: | A B | | B D | sin ∠ D A B = | A C | | C D | sin ∠ D A C , {\displaystyle {{\frac {|AB|}{|BD|}}\sin \angle DAB={\frac {|AC|}{|CD|}}\sin \angle DAC},} which rearranges to the "generalized" version of the theorem. Let D be a point on the line BC, not equal to B or C and such that AD is not an altitude of triangle △ABC. Let B1 be the base (foot) of the altitude in the triangle △ABD through B and let C1 be the base of the altitude in the triangle △ACD through C. Then, if D is strictly between B and C, one and only one of B1 or C1 lies inside △ABC and it can be assumed without loss of generality that B1 does. This case is depicted in the adjacent diagram. If D lies outside of segment BC, then neither B1 nor C1 lies inside the triangle. ∠ DB1B, ∠ DC1C are right angles, while the angles ∠ B1DB, ∠ C1DC are congruent if D lies on the segment BC (that is, between B and C) and they are identical in the other cases being considered, so the triangles △DB1B, △DC1C are similar (AAA), which implies that: | B D | | C D | = | B B 1 | | C C 1 | = | A B | sin ∠ B A D | A C | sin ∠ C A D . {\displaystyle {\frac {|BD|}{|CD|}}={\frac {|BB_{1}|}{|CC_{1}|}}={\frac {|AB|\sin \angle BAD}{|AC|\sin \angle CAD}}.} If D is the foot of an altitude, then, | B D | | A B | = sin ∠ B A D and | C D | | A C | = sin ∠ D A C , {\displaystyle {\frac {|BD|}{|AB|}}=\sin \angle \ BAD{\text{ and }}{\frac {|CD|}{|AC|}}=\sin \angle \ DAC,} and the generalized form follows. α = ∠ B A C 2 = ∠ B A D = ∠ C A D {\textstyle \alpha ={\frac {\angle BAC}{2}}=\angle BAD=\angle CAD} A quick proof can be obtained by looking at the ratio of the areas of the two triangles △BAD, △CAD, which are created by the angle bisector in A. Computing those areas twice using different formulas, that is 1 2 g h {\displaystyle {\tfrac {1}{2}}gh} with base g {\displaystyle g} and altitude h and 1 2 a b sin (γ) {\displaystyle {\tfrac {1}{2}}ab\sin(\gamma)} with sides a, b and their enclosed angle γ, will yield the desired result. Let h denote the height of the triangles on base BC and α {\displaystyle \alpha } be half of the angle in A. Then | △ A B D | | △ A C D | = 1 2 | B D | h 1 2 | C D | h = | B D | | C D | {\displaystyle {\frac {|\triangle ABD|}{|\triangle ACD|}}={\frac {{\frac {1}{2}}|BD|h}{{\frac {1}{2}}|CD|h}}={\frac {|BD|}{|CD|}}} and | △ A B D | | △ A C D | = 1 2 | A B | | A D | sin (α) 1 2 | A C | | A D | sin (α) = | A B | | A C | {\displaystyle {\frac {|\triangle ABD|}{|\triangle ACD|}}={\frac {{\frac {1}{2}}|AB||AD|\sin(\alpha)}{{\frac {1}{2}}|AC||AD|\sin(\alpha)}}={\frac {|AB|}{|AC|}}} yields | B D | | C D | = | A B | | A C | . {\displaystyle {\frac {|BD|}{|CD|}}={\frac {|AB|}{|AC|}}.} Diagram of Stewart's theorem The length of the angle bisector d {\displaystyle d} can be found by d 2 = b c − m n = m n (k 2 − 1) = b c (1 − 1 k 2) {\textstyle d^{2}=bc-mn=mn(k^{2}-1)=bc\left(1-{\frac {1}{k^{2}}}\right)} , where k = b n = c m = b + c a {\displaystyle k={\frac {b}{n}}={\frac {c}{m}}={\frac {b+c}{a}}} is the constant of proportionality from the angle bisector theorem. Proof: By Stewart's theorem (which is more general than Apollonius's theorem), we have b 2 m + c 2 n = a (d 2 + m n) (k n) 2 m + (k m) 2 n = a (d 2 + m n) k 2 (m + n) m n = (m + n) (d 2 + m n) k 2 m n = d 2 + m n (k 2 − 1) m n = d 2 {\displaystyle {\begin{aligned}b^{2}m+c^{2}n&=a(d^{2}+mn)\\(km)^{2}m+(km)^{2}n&=a(d^{2}+mn)\\k^{2}(m+n)mn&=(m+n)(d^{2}+mn)\\k^{2}mn&=d^{2}+mn\\(k^{2}-1)mn&=d^{2}\\\end{aligned}}} exterior angle bisectors (dotted red): Points D, E, F are collinear and the following equations for ratios hold: | E B | | E C | = | A B | | A C | {\displaystyle {\tfrac {|EB|}{|EC|}}={\tfrac {|AB|}{|AC|}}} , | F B | | F A | = | C B | | C A | {\displaystyle {\tfrac {|FB|}{|FA|}}={\tfrac {|CB|}{|CA|}}} , | D A | | D C | = | B A | | B C | {\displaystyle {\tfrac {|DA|}{|DC|}}={\tfrac {|BA|}{|BC|}}} For the exterior angle bisectors in a non-equilateral triangle there exist similar equations for the ratios of the lengths of triangle sides. More precisely if the exterior angle bisector in A intersects the extended side BC in E, the exterior angle bisector in B intersects the extended side AC in D and the exterior angle bisector in C intersects the extended side AB in F, then the following equations hold:[1] | E B | | E C | = | A B | | A C | {\displaystyle {\frac {|EB|}{|EC|}}={\frac {|AB|}{|AC|}}} , | F B | | F A | = | C B | | C A | {\displaystyle {\frac {|FB|}{|FA|}}={\frac {|CB|}{|CA|}}} , | D A | | D C | = | B A | | B C | {\displaystyle {\frac {|DA|}{|DC|}}={\frac {|BA|}{|BC|}}} The three points of intersection between the exterior angle bisectors and the extended triangle sides D, E, F are collinear, that is they lie on a common line.[2] The angle bisector theorem appears as Proposition 3 of Book VI in Euclid's Elements. According to Heath (1956, p. 197 (vol. 2)), the corresponding statement for an external angle bisector was given by Robert Simson who noted that Pappus assumed this result without proof. Heath goes on to say that Augustus De Morgan proposed that the two statements should be combined as follows:[3] If an angle of a triangle is bisected internally or externally by a straight line which cuts the opposite side or the opposite side produced, the segments of that side will have the same ratio as the other sides of the triangle; and, if a side of a triangle be divided internally or externally so that its segments have the same ratio as the other sides of the triangle, the straight line drawn from the point of section to the angular point which is opposite to the first mentioned side will bisect the interior or exterior angle at that angular point. This section needs expansion with: more theorems/results. You can help by adding to it. (September 2020) This theorem has been used to prove the following theorems/results: Coordinates of the incenter of a triangle Circles of Apollonius ^ Alfred S. Posamentier: Advanced Euclidean Geometry: Excursions for Students and Teachers. Springer, 2002, ISBN 9781930190856, pp. 3–4. ^ Roger A. Johnson: Advanced Euclidean Geometry. Dover 2007, ISBN 978-0-486-46237-0, p. 149 (original publication 1929 with Houghton Mifflin Company (Boston) as Modern Geometry). ^ Heath, Thomas L. (1956). The Thirteen Books of Euclid's Elements (2nd ed. [Facsimile. Original publication: Cambridge University Press, 1925] ed.). New York: Dover Publications. (3 vols.): ISBN 0-486-60088-2 (vol. 1), ISBN 0-486-60089-0 (vol. 2), ISBN 0-486-60090-4 (vol. 3). Heath's authoritative translation plus extensive historical research and detailed commentary throughout the text. G. W. I. S. Amarasinghe: On the Standard Lengths of Angle Bisectors and the Angle Bisector Theorem, Global Journal of Advanced Research on Classical and Modern Geometries, Vol. 01(01), pp. 15–27, 2012. A Property of Angle Bisectors at cut-the-knot Intro to angle bisector theorem at Khan Academy Retrieved from " Angle bisector theorem states that an angle bisector of a triangle divides the opposite side into two segments that are proportional to the other two sides of the triangle. An angle bisector is a ray that divides a given angle into two angles of equal measures. Let us learn more about the angle bisector theorem in this article. What is Angle Bisector Theorem? The triangle angle bisector theorem states that in a triangle, the angle bisector of any angle will divide the opposite side in the ratio of the sides containing the angle. Consider the figure below: Here, PS is the bisector of ∠P. According to the angle bisector theorem, PQ/PR = QS/RS or a/b = x/y. An angle bisector is a line or ray that divides an angle in a triangle into two equal measures. The main properties of an angle bisector are that any point on the bisector of an angle is equidistant from the sides of the angle and the angle bisector divides the opposite side of a triangle in the ratio of the adjacent sides, which is known as the angle bisector property of triangle. Angle Bisector Theorem Proof Statement: In a triangle, the angle bisector of any angle will divide the opposite side in the ratio of the sides containing the angle. Let us see the proof of this. Draw a ray CX parallel to AD, and extend BA to intersect this ray at E. By the basic proportionality theorem, we have that if a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, the other two sides are divided in the same ratio. In ΔCBE, DA is parallel to CE. BD/DC = BA/AE ⋯ (1) Now, we are left with proving that AE = AC. Let's mark the angles in the above figure. Since DA is parallel to CE, we have ∠DAB = ∠CEA (corresponding angles) ----- (2) ∠DAC = ∠ACE (alternate interior angles) ----- (3) Since AD is the bisector of ∠BAC, we have ∠DAB = ∠DAC ---- (4). From (2), (3), and (4), we can say that ∠CEA = ∠ACE. It makes ΔACE an isosceles triangle. Since sides opposite to equal angles are equal, we have AC = AE. Substitute AC for AE in equation (1). BD/DC = BA/AC Hence proved. Converse of Angle Bisector Theorem The converse of angle bisector theorem states that if the sides of a triangle satisfy the following condition "If a line drawn from a vertex of a triangle divides the opposite side into two parts such that they are proportional to the other two sides of the triangle", it implies that the point on the opposite side of that angle lies on its angle bisector. Here, it is known to us that sides are in proportion, and from this, we came to a conclusion that the line/ray/segment is the angle bisector of the respective angle. This is known as the converse of angle bisector theorem in geometry. Look at the image below to understand it visually. Angle Bisector Theorem Formula Triangle angle bisector theorem states that "In a triangle, the angle bisector of any angle will divide the opposite side in the ratio of the sides containing the angle". From this, we can write the triangle angle bisector theorem formula as, \(\dfrac{\text{BD}}{\text{DC}} = \dfrac{\text{AB}}{\text{AC}}\) ► Related Topics Listed below are a few interesting topics related to the triangle angle bisector theorem, take a look. Example 1: Amy drew a triangle ABC on the board where AD is the line drawn on side BC, where, AB = 4 in, AC = 6 in, BD = 1.6 in, and DC = 2.4 in. She wants to know whether AD is the angle bisector of ∠A. Can you help her? Solution: To show whether AD is the angle bisector or not, let us use the angle bisector theorem. So, we need to prove that BD/DC = AB/AC. Let's find the ratio AB/AC. AB/AC = 4/6 = 2/3 Let's find the ratio BD/DC. BD/DC = 1.6/2.4 = 2/3 Both the ratios are equal. Therefore, in the triangle drawn by Amy, AD bisects ∠A. Example 2: In ΔXYZ, XE is the bisector of ∠X. Let XY = 4 units, YE = 2 units, and EZ = 3 units. Can you find the length of XZ? Solution: Given that, XE is the bisector of ∠X. According to the angle bisector theorem formula, YE/EZ = XY/XZ 2/3 = 4/XZ XZ = 4/2 × 3 XZ = 6 Therefore, the length of XZ = 6 units. Example 3: Look at ΔABC shown below. If BD bisects ∠B, can you find the value of x? Solution: Given that, BD is the bisector of ∠B. According to the triangle angle bisector theorem, AB/BC = AD/DC x/(x-2) = (x+2)/(x-1) x(x−1) = (x−2)(x+2) x2 − x = x2 − 4 −x = −4 x = 4 Therefore, the value of x is 4. View Answer > go to slidego to slidego to slide Breakdown tough concepts through simple visuals. Math will no longer be a tough subject, especially when you understand the concepts through visualizations. Book a Free Trial Class FAQs on Angle Bisector Theorem The triangle angle bisector theorem states that "The bisector of any angle inside a triangle divides the opposite side into two parts proportional to the other two sides of the triangle which contain the angle." What is the Formula for Angle Bisector Theorem? Let AD be the bisector of ∠A in ΔABC. According to the angle bisector theorem formula, BD/DC = AB/AC. How are the Side-Splitter Theorem and the Angle Bisector Theorem Similar? The only similarity between the side-splitter theorem and the angle bisector theorem is that both the theorems are related to the proportions of side lengths of the triangle. How to Use Angle Bisector Theorem? The triangle angle bisector theorem can be used to find the missing lengths of the sides of a triangle. It establishes a relation between the sides. What is the Converse of the Angle Bisector Theorem? If a line or a ray AD is drawn in ΔABC such that BD/DC = AB/AC, then AD bisects the ∠A. This is the statement of angle bisector theorem converse. How to Prove Angle Bisector Theorem? To prove the angle bisector theorem, we need to extend the sides of the triangle and make another triangle right next to it. Then, we use the basic proportionality theorem to state the relationship between the sides of the triangle drawn.