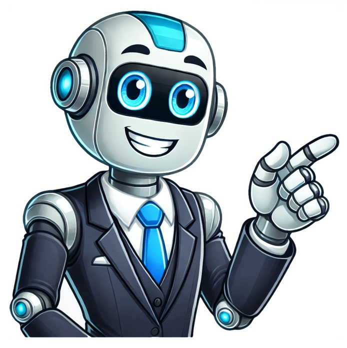

On the complex numbers polar form page, we see examples of converting from complex number cartesian form to complex number polar form. CARTESIAN FORM: z = a + bi POLAR FORM: z = r(cosθ + isinθ) Converting the other way from polar form to complex number cartesian form is also possible. To see this in action, we can look at examples (1.1) and (1.2) from the complex numbers polar form page. (1.1) Polar Form: 2√2(cos(\bf{\frac{\pi}{4}}) + isin(\bf{\frac{\pi}{4}})) To convert to Cartesian Form. We can multiply out as it sits, using the exact values for the cos and sin. z = (2√2 × cos(\bf{\frac{\pi}{4}})) + i(2√2 × sin(\bf{\frac{\pi}{4}})) z = (2√2 × \bf{\frac{1}{\sqrt{2}}}) + i(2√2 × \bf{\frac{1}{\sqrt{2}}}) z = 2 + i2 (1.2) Polar Form: √17(cos(1.816) + isin(1.816)) Now as the radians here are rounded to 3 decimal places, the initial calculations will be slightly out. But the numbers obtained will be close enough to round both down and up to the given original values in this example from the polar form page. For Cartesian Form: z = (√17 × cos(1.816)) + i(√17 × sin(1.816)) z = –1.0009... + i3.9997... Which can be rounded to. z = –1 + i4 It should also be mentioned that a complex number can also be expressed in “Exponential Form”. We can also convert from and to this exponential form. If we observe Euler’s Formula. eiθ = cosθ + isinθ A complex number can written as   z = reiθ. Where r is once again the modulus of the complex number. With θ again being the argument, specifically in radians. Examples   (2.1) Write in exponential form   z = 1 + i√3. Solution   r = \bf{\sqrt{1^2 \space + \space (\sqrt{3})^2}} = 2 a > 0 , b > 0   =>   θ = tan-1(\bf{\frac{\sqrt{3}}{1}}) = \bf{\frac{\pi}{3}} In exponential form: z = 2ei\bf{\frac{\pi}{3}} (2.2) Write in exponential form z = 2√2(cos(\bf{\frac{\pi}{4}}) + isin(\bf{\frac{\pi}{4}})). Solution   In exponential form: z = 2√2ei\bf{\frac{\pi}{4}} (2.3) Write in cartesian form   z = 4ei\bf{\frac{\pi}{6}}. Solution   Put the complex number into polar form first. z = 4(cos(\bf{\frac{\pi}{6}}) + isin(\bf{\frac{\pi}{6}})) Then carry out the calculation. z = (4 × cos(\bf{\frac{\pi}{6}})) + i(4 × sin(\bf{\frac{\pi}{6}})) z = (4 x \bf{\frac{\sqrt{3}}{2}}) + i(4 × \bf{\frac{1}{2}}) z = 2√3 + i2 Home › Algebra 2 › Complex Number Converting Return to TOP of page Show Mobile Notice Show All Notes Hide All Notes Mobile Notice You appear to be on a device with a "narrow" screen width (i.e. you are probably on a mobile phone). Due to the nature of the mathematics on this site it is best viewed in landscape mode. If your device is not in landscape mode many of the equations will run off the side of your device (you should be able to scroll/swipe to see them) and some of the menu items will be cut off due to the narrow screen width. Most people are familiar with complex numbers in the form \(z = a + bi\), however there are some alternate forms that are useful at times. In this section we’ll look at both of those as well as a couple of nice facts that arise from them. Geometric Interpretation Before we get into the alternate forms we should first take a very brief look at a natural geometric interpretation of complex numbers since this will lead us into our first alternate form. Consider the complex number \(z = a + bi\). We can think of this complex number as either the point \(\left({a,b} \right)\) in the standard Cartesian coordinate system or as the vector that starts at the origin and ends at the point \(\left({a,b} \right)\). An example of this is shown in the figure below. In this interpretation we call the \(x\)-axis the real axis and the \(y\)-axis the imaginary axis. We often call the \(xy\)-plane in this interpretation the complex plane. Note as well that we can now get a geometric interpretation of the modulus. From the image above, we can see that \(\left| z \right| = \sqrt {{a^2} + {b^2}} \) is nothing more than the length of the vector that we’re using to represent the complex number \(z = a + bi\). This interpretation also tells us that the inequality \(\left| {{z_1}} \right| < \left| {{z_2}} \right|\) means that \({z_1}\) is closer to the origin (in the complex plane) than \({z_2}\) is. Polar Form Let’s now take a look at the first alternate form for a complex number. If we think of the non-zero complex number \(z = a + bi\) as the point \(\left({a,b} \right)\) in the \(xy\)-plane we also know that we can represent this point by the polar coordinates \(\left({r,\theta } \right)\), where \(r\) is the distance of the point from the origin and \(\theta \) is the angle, in radians, from the positive \(x\)-axis to the ray connecting the origin to the point. When working with complex numbers we assume that \(r\) is positive and that \(\theta \) can be any of the possible (both positive and negative) angles that end at the ray. Note that this means that there are literally an infinite number of choices for \(\theta \). We excluded \(z = 0\) since \(\theta \) is not defined for the point (0,0). We will therefore only consider the polar form of non-zero complex numbers. We have the following conversion formulas for converting the polar coordinates \(\left({r,\theta } \right)\) into the corresponding Cartesian coordinates of the point, \(\left({a,b} \right)\). \[a = r\cos \theta \hspace{0.75in} b = r\sin \theta \] If we substitute these into \(z = a + bi\) and factor an \(r\) out we arrive at the polar form of the complex number, \begin{equation}z = r\left({\cos \theta + i\sin \theta } \right) \label{eq:eq1}\end{equation} Note as well that we also have the following formula from polar coordinates relating \(r\) to \(a\) and \(b\). \[r = \sqrt {{a^2} + {b^2}} \] but, the right side is nothing more than the definition of the modulus and we see that, \begin{equation}r = \left| z \right|\label{eq:eq2} \end{equation} So, sometimes the polar form will be written as, \begin{equation}z = \left| z \right|\left({\cos \theta + i\sin \theta } \right)\label{eq:eq3}\end{equation} The angle \(\theta \) is called the argument of \(z\) and is denoted by, \[\theta = \arg z\] The argument of \(z\) can be any of the infinite possible values of \(\theta \) each of which can be found by solving \begin{equation}\tan \theta = \frac{b}{a}\label{eq:eq4}\end{equation} and making sure that \(\theta \) is in the correct quadrant. Note as well that any two values of the argument will differ from each other by an integer multiple of \(2\pi \). This makes sense when you consider the following. For a given complex number \(z\) pick any of the possible values of the argument, say \(\theta \). If you now increase the value of \(\theta \), which is really just increasing the angle that the point makes with the positive \(x\)-axis, you are rotating the point about the origin in a counter-clockwise manner. Since it takes \(2\pi \) radians to make one complete revolution you will be back at your initial starting point when you reach \(\theta + 2\pi \) and so have a new value of the argument. See the figure below. If you keep increasing the angle you will again be back at the starting point when you reach \(\theta + 4\pi \), which is again a new value of the argument. Continuing in this fashion we can see that every time we reach a new value of the argument we will simply be adding multiples of \(2\pi \) onto the original value of the argument. Likewise, if you start at \(\theta \) and decrease the angle you will be rotating the point about the origin in a clockwise manner and will return to your original starting point when you reach \(\theta - 2\pi \). Continuing in this fashion and we can again see that each new value of the argument will be found by subtracting a multiple of \(2\pi \) from the original value of the argument. So, we can see that if \({\theta _1}\) and \({\theta _2}\) are two values of \(\arg z\) then for some integer \(k\) we will have, \begin{equation}{\theta _1} - {\theta _2} = 2\pi k\label{eq:eq5}\end{equation} Note that we’ve also shown here that \(z = r\left({\cos \theta + i\sin \theta } \right)\) is a parametric representation for a circle of radius \(r\) centered at the origin and that it will trace out a complete circle in a counter-clockwise direction as the angle increases from \(\theta \) to \(\theta + 2\pi \). The principal value of the argument (sometimes called the principal argument) is the unique value of the argument that is in the range \(- \pi < \arg z \le \pi \) and is denoted by \({\mathop{\rm Arg}olimits} z\). Note that the inequalities at either end of the range tells that a negative real number will have a principal value of the argument of \({\mathop{\rm Arg}olimits} z = \pi \). Recalling that we noted above that any two values of the argument will differ from each other by a multiple of \(2\pi \) leads us to the following fact. \begin{equation}\arg z = {\mathop{\rm Arg}olimits} z + 2\pi n \hspace{0.25in} n = 0, \pm 1, \pm 2, \ldots \label{eq:eq:6}\end{equation} We should probably do a couple of quick numerical examples at this point before we move on to look the second alternate form of a complex number. Example 1 Write down the polar form of each of the following complex numbers. \(z = - 1 + i\,\sqrt 3 \) \(z = - 9 \) \(z = 12\,i\) Show All Solutions Hide All Solutions a \(z = - 1 + i\,\sqrt 3 \) Show Solution Let’s first get \(r\). \[r = \left| z \right| = \sqrt {1 + 3} = 2\] Now let’s find the argument of \(z\). This can be any angle that satisfies \(\eqref{eq:eq4}\), but it’s usually easiest to find the principal value so we’ll find that one. The principal value of the argument will be the value of \(\theta \) that is in the range \(- \pi < \theta \le \pi \), satisfies, \[\tan \theta = \frac{{\sqrt 3 }}{{ - 1}} \hspace{0.25in} \Rightarrow \hspace{0.25in} \theta = {\tan ^{ - 1}}\left({ - \sqrt 3 } \right)\] and is in the second quadrant since that is the location the complex number in the complex plane. If you’re using a calculator to find the value of this inverse tangent make sure that you understand that your calculator will only return values in the range \(- \frac{\pi }{2} < \theta < \frac{\pi }{2}\) and so you may get the incorrect value. Recall that if your calculator returns a value of \({\theta _1}\) then the second value that will also satisfy the equation will be \({\theta _2} = {\theta _1} + \pi \). So, if you’re using a calculator be careful. You will need to compute both and the determine which falls into the correct quadrant to match the complex number we have because only one of them will be in the correct quadrant. In our case the two values are, \[{\theta _1} = - \frac{\pi }{3} \hspace{0.25in} {\theta _2} = - \frac{\pi }{3} + \pi = \frac{{2\pi }}{3}\] The first one is in quadrant four and the second one is in quadrant two and so is the one that we’re after. Therefore, the principal value of the argument is, \[{\mathop{\rm Arg}olimits} \,z = \frac{{2\pi }}{3}\] and all possible values of the argument are then \[\arg z = \frac{{2\pi }}{3} + 2\pi n \hspace{0.25in} n = 0, \pm 1, \pm 2, \ldots \] Now, let’s actually do what we were originally asked to do. Here is the polar form of \(z = - 1 + i\,\sqrt 3 \). \[z = 2\left({\cos \left({\frac{{2\pi }}{3}} \right) + i\sin \left({\frac{{2\pi }}{3}} \right)} \right)\] Now, for the sake of completeness we should acknowledge that there are many more equally valid polar forms for this complex number. To get any of the other forms we just need to compute a different value of the argument by picking \(n\). Here are a couple of other possible polar forms. \begin{align*}z & = 2\left({\cos \left({\frac{{8\pi }}{3}} \right) + i\sin \left({\frac{{8\pi }}{3}} \right)} \right) & \hspace{0.25in} & n = 1\\ z & = 2\left({\cos \left({ - \frac{{16\pi }}{3}} \right) + i\sin \left({ - \frac{{16\pi }}{3}} \right)} \right) & \hspace{0.25in} & n = - 3\end{align*} b \(z = - 9 \) Show Solution In this case we’ve already noted that the principal value of a negative real number is \(\pi \) so we don’t need to compute that. For completeness sake here are all possible values of the argument of any negative number. \[\arg z = \pi + 2\pi n = \pi \left({1 + 2n} \right) \hspace{0.25in} n = 0, \pm 1, \pm 2, \ldots \] Now, \(r\) is, \[r = \left| z \right| = \sqrt {81 + 0} = 9\] The polar form (using the principal value) is, \[z = 9\left({\cos \left(\pi \right) + i\sin \left(\pi \right)} \right)\] Note that if we’d had a positive real number the principal value would be \({\mathop{\rm Arg}olimits}\, z = 0\) c \(z = 12\,i\) Show Solution This another special case much like real numbers. If we were to use \(\eqref{eq:eq4}\) to find the argument we would run into problems since the real part is zero and this would give division by zero. However, all we need to do to get the argument is think about where this complex number is in the complex plane. In the complex plane purely imaginary numbers are either on the positive \(y\)-axis or the negative \(y\)-axis depending on the sign of the imaginary part. For our case the imaginary part is positive and so this complex number will be on the positive \(y\)-axis. Therefore, the principal value and the general argument for this complex number is, \[{\mathop{\rm Arg}olimits} z = \frac{\pi }{2} \hspace{0.5in} \arg z = \frac{\pi }{2} + 2\pi n = \pi \left({\frac{1}{2} + 2n} \right) \hspace{0.25in} n = 0, \pm 1, \pm 2, \ldots \] Also, in this case \(r\) = 12 and so the polar form (again using the principal value) is, \[z = 12\left({\cos \left({\frac{\pi }{2}} \right) + i\sin \left({\frac{\pi }{2}} \right)} \right)\] Exponential Form Now that we’ve discussed the polar form of a complex number we can introduce the second alternate form of a complex number. First, we’ll need Euler’s formula, \begin{equation}{{\bf{e}}^{i\,\theta }} = \cos \theta + i\sin \theta \label{eq:eq7}\end{equation} With Euler’s formula we can rewrite the polar form of a complex number into its exponential form as follows. \[z = r{{\bf{e}}^{i\,\theta }}\] where \(\theta = \arg z\) and so we can see that, much like the polar form, there are an infinite number of possible exponential forms for a given complex number. Also, because any two arguments for a give complex number differ by an integer multiple of \(2\pi \) we will sometimes write the exponential form as, \[z = r{{\bf{e}}^{i\,\left({\theta + 2\pi n} \right)}} \hspace{0.25in} n = 0, \pm 1, \pm 2, \ldots \] where \(\theta \) is any value of the argument although it is more often than not the principal value of the argument. To get the value of \(r\) we can either use \(\eqref{eq:eq3}\) to write the exponential form or we can take a more direct approach. Let’s take the direct approach. Take the modulus of both sides and then do a little simplification as follows, \[\left| z \right| = \left| {r{{\bf{e}}^{i\,\theta }}} \right| = \left| r \right|\,\left| {{{\bf{e}}^{i\,\theta }}} \right| = \left| r \right|\,\left| {\cos \theta + i\sin \theta } \right| = \sqrt {{r^2} + 0} \,\,\sqrt {{{\cos }^2}\theta + {{\sin }^2}\theta } = r\] and we see that \(r = \left| z \right|\). Note as well that because we can consider \(z = r\left({\cos \theta + i\sin \theta } \right)\) as a parametric representation of a circle of radius \(r\) and the exponential form of a complex number is really another way of writing the polar form we can also consider \(z = r{{\bf{e}}^{i\,\theta }}\) a parametric representation of a circle of radius \(r\). Now that we’ve got the exponential form of a complex number out of the way we can use this along with basic exponent properties to derive some nice facts about complex numbers and their arguments. First, let’s start with the non-zero complex number \(z = r{{\bf{e}}^{i\,\theta }}\). In the arithmetic section we gave a fairly complex formula for the multiplicative inverse, however, with the exponential form of the complex number we can get a much nicer formula for the multiplicative inverse. \[{z^{ - 1}} = {\left({r{{\bf{e}}^{i\,\theta }}} \right)^{ - 1}} = {r^{ - 1}}{\left({{{\bf{e}}^{i\,\theta }}} \right)^{ - 1}} = {r^{ - 1}}{{\bf{e}}^{ - i\,\theta }} = \frac{1}{r}{{\bf{e}}^{i\,\left({ - \theta } \right)}}\] Note that since \(r\) is a non-zero real number we know that \({r^{ - 1}} = \frac{1}{r}\). So, putting this together the exponential form of the multiplicative inverse is, \begin{equation}{z^{ - 1}} = \frac{1}{r}{{\bf{e}}^{i\,\left({ - \theta } \right)}}\label{eq:eq8}\end{equation} and the polar form of the multiplicative inverse is, \begin{equation}{z^{ - 1}} = \frac{1}{r}\left({\cos \left({ - \theta } \right) + i\sin \left({ - \theta } \right)} \right)\label{eq:eq9}\end{equation} We can also get some nice formulas for the product or quotient of complex numbers. Given two complex numbers \({z_1} = {r_1}\,{{\bf{e}}^{i\,{\theta _{\,1}}}}\) and \({z_2} = {r_2}\,{{\bf{e}}^{i\,{\theta _{\,2}}}}\), where \({\theta _1}\) is any value of \(\arg {z_1}\) and \({\theta _2}\) is any value of \(\arg {z_2}\), we have \begin{align}{z_1}{z_2} &= \left({{r_1}\,{{\bf{e}}^{i\,{\theta _{\,1}}}}} \right)\left({{r_2}\,{{\bf{e}}^{i\,{\theta _{\,2}}}}} \right) = {r_1}\,{r_2}{{\bf{e}}^{i\,\left({{\theta _{\,1}} + {\theta _{\,2}}} \right)}}\label{eq:eq10}\\ & onumber \\ \frac{{{z_1}}}{{{z_2}}} &= \frac{{{r_1}\,{{\bf{e}}^{i\,{\theta _{\,1}}}}}}{{{r_2}\,{{\bf{e}}^{i\,{\theta _{\,2}}}}}} = \frac{{{r_1}}}{{{r_2}}}{{\bf{e}}^{i\,\left({{\theta _{\,1}}\, - \,{\theta _{\,2}}} \right)}}\label{eq:eq11}\end{align} The polar forms for both of these are, \begin{align}{z_1}{z_2} & = {r_1}\,{r_2}\left({\cos \left({{\theta _{\,1}} + {\theta _{\,2}}} \right) + i\sin \left({{\theta _{\,1}} + {\theta _{\,2}}} \right)} \right)\label{eq:eq12} \\ & onumber \\ \frac{{{z_1}}}{{{z_2}}} & = \frac{{{r_1}}}{{{r_2}}}\left({\cos \left({{\theta _{\,1}}\, - \,{\theta _{\,2}}} \right) + i\sin \left({{\theta _{\,1}}\, - \,{\theta _{\,2}}} \right)} \right)\label{eq:eq13}\end{align} We can also use \(\eqref{eq:eq10}\) and \(\eqref{eq:eq11}\) to get some nice facts about the arguments of a product and a quotient of complex numbers. Since \({\theta _1}\) is any value of \(\arg {z_1}\) and \({\theta _2}\) is any value of \(\arg {z_2}\) we can see that, \begin{align}\arg \left({{z_1}{z_2}} \right) & = \arg {z_1} + \arg {z_2}\label{eq:eq14} \\ & onumber \\ \arg \left({\frac{{{z_1}}}{{{z_2}}}} \right) & = \arg {z_1} - \arg {z_2}\label{eq:eq15} \end{align} Note that \(\eqref{eq:eq14}\) and \(\eqref{eq:eq15}\) may or may not work if you use the principal value of the argument, \({\rm{Arg }}\,z\). For example, consider \({z_1} = i\) and \({z_2} = - 1\). In this case we have \({z_1}{z_2} = - i\) and the principal value of the argument for each is, \[{\mathop{\rm Arg}olimits} \left(i \right) = \frac{\pi }{2} \hspace{0.5in} {\mathop{\rm Arg}olimits} \left({ - 1} \right) = \pi \hspace{0.5in} {\mathop{\rm Arg}olimits} \left({ - i} \right) = - \frac{\pi }{2}\] However, \[{\mathop{\rm Arg}olimits} \left(i \right) + {\mathop{\rm Arg}olimits} \left({ - 1} \right) = \frac{{3\pi }}{2} e - \frac{\pi }{2}\] and so \(\eqref{eq:eq14}\) doesn’t hold if we use the principal value of the argument. Note however, if we use, \[\arg \left(i \right) = \frac{\pi }{2} \hspace{0.5in}\arg \left({ - 1} \right) = \pi \] then, \[\arg \left(i \right) + \arg \left({ - 1} \right) = \frac{{3\pi }}{2}\] is valid since \(\frac{{3\pi }}{2}\) is a possible argument for –\(i\), it just isn’t the principal value of the argument. As an interesting side note, \(\eqref{eq:eq15}\) actually does work for this example if we use the principal arguments. That won’t always happen, but it does in this case so be careful! We will close this section with a nice fact about the equality of two complex numbers that we will make heavy use of in the next section. Suppose that we have two complex numbers given by their exponential forms, \({z_1} = {r_1}\,{{\bf{e}}^{i\,{\theta _{\,1}}}}\) and \({z_2} = {r_2}\,{{\bf{e}}^{i\,{\theta _{\,2}}}}\). Also suppose that we know that \({z_1} = {z_2}\). In this case we have, \[{r_1}\,{{\bf{e}}^{i\,{\theta _{\,1}}}} = {r_2}\,{{\bf{e}}^{i\,{\theta _{\,2}}}}\] Then we will have \({z_1} = {z_2}\) if and only if, \begin{equation}{r_1} = {r_2} \hspace{0.25in} {\rm{and}} \hspace{0.25in} {\theta _2} = {\theta _1} + 2\pi k\,\,\,{\mbox{for some integer }}k{\rm{ }}\left({i.e.\,\,k = 0, \pm 1, \pm 2, \ldots } \right)\label{eq:eq16}\end{equation} Note that the phrase “if and only if” is a fancy mathematical phrase that means that if \({z_1} = {z_2}\) is true then so is \(\eqref{eq:eq16}\) and likewise, if \(\eqref{eq:eq16}\) is true then we’ll have \({z_1} = {z_2}\). This may seem like a silly fact, but we are going to use this in the next section to help us find the powers and roots of complex numbers. The complex number is said to be in Cartesian form. There are, however, other ways to write a complex number, such as in modulus-argument (polar) form.How do I write a complex number in modulus-argument (polar) form?The Cartesian form of a complex number, , is written in terms of its real part, , and its imaginary part, If we let and , then it is possible to write a complex number in terms of its modulus, , and its argument, , called the modulus-argument (polar) form, given by...It is usual to give arguments in the range Negative arguments should be shown clearly, e.g. without simplifying   to either or Occasionally you could be asked to give arguments in the range If a complex number is given in the form , then it is not currently in modulus-argument (polar) form due to the minus sign, but can be converted as follows...By considering transformations of trigonometric functions, we see that and Therefore can be written as , now in the correct form and indicating an argument of To convert from modulus-argument (polar) form back to Cartesian form, evaluate the real and imaginary partsE.g. becomes Write in the form where and are exact.When two complex numbers, and , are multiplied to give , their moduli are also multipliedWhen two complex numbers, and , are divided to give , their moduli are also dividedWhen two complex numbers, and , are multiplied to give , their arguments are addedWhen two complex numbers, and , are divided to give , their arguments are subtractedThe main benefit of writing complex numbers in modulus-argument (polar) form is that they multiply and divide very easily (often quicker than when in Cartesian form)To multiply two complex numbers, and , in modulus-argument (polar) form we use the rules from above to multiply their moduli and add their argumentsSo if and then the rules above give...Sometimes the new argument, , does not lie in the range (or   if this is being used)An out-of-range argument can be adjusted by either adding or subtracting E.g. If and   then   This is currently not in the range , but by subtracting from to give , a new argument is formed that lies in the correct range and represents the same angle on an Argand diagramThe rules of multiplying the moduli and adding the arguments can also be applied when......multiplying three complex numbers together, , or more...finding powers of a complex number (e.g. can be written as)Whilst not examinable, the rules for multiplication can be proved algebraically by multiplying by , expanding the brackets and using compound angle formulaeTo divide two complex numbers, and in modulus-argument (polar) form, we use the rules from above to divide their moduli and subtract their argumentsSo if and then the rules above give...As with multiplication, sometimes the new argument, , can lie out of the range (or the range if this is being used)You can add or subtract to bring out-of-range arguments back in rangeWhilst not examinable, the rules for division can be proved algebraically by dividing by , using complex division and compound angle formulaeLet and a)Find , giving your answer in the form where b) Find , giving your answer in the form where Did this page help you? Summary Complex numbers can be represented in cartesian form (a + bi) or in polar form (r*e^(i * theta)). The magnitude of a complex number is found by multiplying by its complex conjugate (a-bi) and then taking the square root of the product. In polar form, r is the magnitude. Imaginary Numbers: What are they? Easy answer: The square root of -1 is represented by the number "i". "i" looks like a variable but it is not; it is the number such that its sqaure equals negative one. The square roots of other negative numbers can be represented in terms of i. For example, the square root of -9 is 3i. Tricky stuff: How a negative number can even have a square boggles the mind. That's why "i" is imaginary, I suppose, but the fact that we defined and use it is funky. Another point: don't all numbers have two square roots? The square root of 9 is 3 and -3. What about -1? It should have two roots as well (since we said that it is allowed to have roots). Maybe -i will work as well (just like -3 works for 9). Well, if we square -i we get (-i) * (-i) = (-1)i * (-1)i = i * i = -1. Another point: imaginary numbers have real squares. Why? When you square a number b*i   (like 3i, or -6i), you get: (bi)*(bi) =(b * b)(i * i) = -(b^2).   So no matter what number b you choose, you get a real result. Fourth powers are another matter, but sticking to squares we are safe. Whoa. So +i and -i have the same square. Why do we choose one over the other? I shall return to this shortly, but for now I will let the excitement build. Complex Numbers We now have two types of numbers: real, which are all the regular numbers you know and love, and imaginary numbers, the numbers that are square roots of negative numbers. It is easy to tell them apart: imaginary numbers have an "i". Real numbers don't. Now, complex numbers are numbers which have two parts: a real part and an imaginary part. One way to write them is like this: a + bi.   "a" is the real part (no i) and "bi" is the imaginary part (has an i). An example of some complex numbers are: 3 + 4i   a=3, b=4 3   a=3, b=0 -6i   a=0, b=-6 As you can see, every number can be written as a complex number. Some numbers, like 5 or -9 don't have imaginary parts, and other numbers, like 3i, don't have real parts. Complex numbers are commonly written in the form z = a +bi   (z is complex, a and b are real). Notice that z is a single number. It has two components, but it is still one number. Think of it in terms of fractions: a fraction is a single number that has two parts: a numerator and a denominator. In general, each component of the fraction is different from the fraction itself.   The same goes for complex numbers: z has two components (1 real and 1 imaginary), and each component alone is (generally) different from z. The components combine to create the complex number z. Graphing Complex Numbers Using a + bi notation, we can even draw complex numbers on the complex plane. We are used to x and y axis: we plotted points as (x,y) pairs. Now, instead of having x and y coordinates, we have real and imaginary coordinates. Notice how the complex numbers can be broken down into (a,b) pairs. 3+4i becomes (3,4) on the complex plane. We draw a vector from the origin (0,0) to the point (a, b) that represents the complex number. 3 +3i looks like this, with imaginary numbers on the vertical and real numbers on the horizontal: This is just like a normal graph, except we have changed the labels on the axes... You probably know that a point can be represented in cartesian or polar coordiantes. Cartesian coordiantes are in the form (x,y) and give the two (or more) components of a point. Polar coordinates use a direction and magnitude, and have points in the form (r, theta). For example, the point (1,1) in cartesian is (2^.5, 45) in polar. 45 represents the direction (45 degrees above the horizontal) and 2^.5 represents the amount of distance to go (thank you Pythagoras). The angles start at zero and go counter-clockwise. To go 1 unit downward: cartesian: (0,-1) polar: (1, 270). To convert between the two: Cartesian: (a, b)   Polar: r = sqrt(a^2 + b^2) [Pythagoran thm], theta = arctan(b/a) Polar: (r, theta) Cartesian: a = r*cos(theta) b = r*sin(theta) You don't have to memorize these by any means. Draw a triangle and you can figure it out (link). It's better to learn the intuition behind a concept and derive it when you need it. Intuition is hard to forget; formulas are easy. [An aside: To express a point in two dimensions you need two peices of data. We are used to the data coming in an (x,y) pair. Now we see it can also be represented as an (r, theta) pair. Are there any more ways to represent a point on a plane? To represent a point in three dimensions, we need three pieces of data. There are a few ways to do this (link).] You will probably seen theta written in terms of radians. Polar coordinates may seem like a hassle: we have to take our complex number and figure out the magnitude and direction. With cartesian coordinates, it is simply (a, b). The next section will justify why we use polar. Incredible Math Relation Ok, I'll admit that very few things in math can be called "exciting". Intersting, maybe (don't roll your eyes) but exciting? This, my friends, is one of those rare moments. I was in hysterics when I first learned of it. The relation is: This formula is just... amazing. It relates e, which is an irrational (infinite decimal places) and funky number to begin with, to i, an imaginary numbers, and also to sine and cosine, which are just regular functions that have rational values . Whoa. To see why it is true, click here. For example, e^(i*pi) = -1. That equation has two irrantional numbers, and somehow the exponential e pops out a negative number. Ok, that's enough blathering about the beauty of that equation, let's see what it can do. Suppose we multiply both sides by some number r. Then we get: Let's look at this for a bit. It is strikingly similar to some of the equations for converting between cartesian and polar coordinates. Indeed, (rcos(theta), rsin(theta)) is the (x,y) pair for a point originally expressed in (r, theta) form.   But the sin has an i term, so the number is complex. Now we have an (a,b) term, with a = rcos(theta) and b = rsin(theta). We have found the polar form for complex numbers. Instead of being an (r, theta) pair we can write any complex number z as: z = a +bi or   z = re^(i*theta) The rules for converting between the two are the same. r = sqrt(a^2 + b^2) and theta = arctan(b/a) If we choose the right numbers for r and theta, then z = a +bi = re^(i * theta).   This is all thanks to the beauty of the above formula. Complex Conjugates and Magnitudes Remember how you were at the edge of your seat wondering why we choose +i instead of -i as the square root of -1? Now we can see where it comes in. The normal method of finding a magnitude is to square a number and then take its square root. For positive real numbers this just gives us the original number, and for negative real numbers (like -9) it will give the absolute value (its magnitude).   Thus, both 9 and -9 have the same magnitude of 9. They are the same distance from the origin, just in different directions. Complex numbers aren't quite so simple. Taking 1 + i as an example, if we try and square this and take the square root we get: magnitude(?) = sqrt((1+i)^2)   = sqrt(1 +2i -1) = sqrt(2i). But we showed earlier that imaginary numbers can't have square roots. (LINK WHY). Uh oh. Complex conjugates save the day. Because our decision to define i as positive was arbitrary, we can't exclude the possibility of a negative i. We define the complex conjugate of (a + bi) as (a - bi). If z is a complex number, its complex conjugate is usually written as z with a bar over it. Now, instead of squaring a complex number then taking its square root, we multiply it by its complex conjugate then take the square root. For any number (a + bi) we get Magnitude = sqrt((a+bi) * (a -bi)) = sqrt(a^2   +abi -abi + b^2) = sqrt(a^2 + b^2). It looks just like the formula for regular cartesian coordinates! (Pythagorean theorem to find lengths). Thus, the magnitude of (3 + 4i) is sqrt(9 + 16) = 5. On a last note, if you want to find the complex conjugate of any complex number, just switch all the i's to "-i". It doesn't matter if they are in exponentials or denominators or inside square roots: just switch them all. For complex numbers in polar form (re^(i*theta)), the magnitude is just r. The polar form of imaginary numbers is useful because multiplication becomes addition when you are dealing with exponentials. This is much, much easier than expanding out loads of cosine and sine terms. Also, you don't have to remember the sine and cosine angle addition formulas; the exponentitals can do it for you. This is very useful when you are analyzing circuits. ← Back to Library A complex number is written in Cartesian form as z = x + iy, where x is the real part, y is the imaginary part, and i denotes the square root of -1. This representation, also known as the rectangular form, allows for intuitive operations, formulating real and imaginary components separately. Using real and imaginary components, Cartesian form straightforwardly handles addition and subtraction of complex numbers. The polar form of a complex number uses the magnitude (r) and the angle (θ) of the complex number in the complex plane: z = r(cos θ + i sin θ). To convert from the Cartesian form to polar form, use the equations: r = sqrt(x^2 + y^2) and θ = tan^-1(y/x). The polar form offers a practical approach while multiplying or dividing complex numbers as it turns into simple multiplication or division of magnitude and addition or subtraction of angles. Note that computer programming languages tend to use the function atan2(y, x) rather than atan(y/x) because atan2 correctly deals with x being zero and gives answers in the correct quadrant. Euler’s form is a concise representation of the polar form: z = re^(iθ). To convert from Cartesian form to Euler’s form, utilise the same conversions as for polar form: r = sqrt(x^2 + y^2) and θ = tan^-1(y/x). Euler’s form is particularly useful in solving problems involving powers and roots of complex numbers, as exponentiation and root extraction become more straightforward operations. The complex conjugate of a complex number in Cartesian form gives significant insights. The complex conjugate of z = x + iy is z̄ = x - iy. Multiplying a complex number by its complex conjugate simplifies the calculation and results in a real number equal to the square of its magnitude: z*z̄ = x^2 + y^2 = r^2. A complex number in Cartesian form can be represented as a 2x2 real matrix, which can be helpful in visualising certain types of transformations. The complex number z = x + iy has the matrix form [x -y; y x]. ← Back to Library A complex number is written in Cartesian form as z = x + iy, where x is the real part, y is the imaginary part, and i denotes the square root of -1. This representation, also known as the rectangular form, allows for intuitive operations, formulating real and imaginary components separately. Using real and imaginary components, Cartesian form straightforwardly handles addition and subtraction of complex numbers. 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